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A relation between the radiation and scattering of surface waves by axisymmetric bodies

By A. M. J. DAVIS
Department of Mathematics, University College London

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Results corresponding to those of Newman (1975) for wholly or partially immersed cylinders are obtained, namely relations between the phase angles of the cylindrical outgoing surface waves generated either by the forced oscillations of an immersed axisymmetric body or by the scattering effect of the body on a plane wave.

1. Introduction

A smooth finite body of arbitrary shape is wholly or partially immersed in incompressible inviscid fluid under gravity. The fluid is unbounded horizontally and may be infinitely deep or have finite, constant depth. Cylindical polar coordinates (R, θ, z) are chosen with z measured vertically downwards and origin in the mean free surface such that a portion of the z axis is interior to the body. Surface tension is neglected and the fluid motion, which is periodic with period $2\pi/\sigma$, is assumed small enough for the equations to be linearized. Then the velocity potential, which is of the form $\mathrm{Re}\,[\phi(R,\theta,z)\,e^{-i\sigma t}]$, satisfies

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{1.1}$$

throughout the fluid region and the boundary conditions

$$K\phi+\partial\phi/\partial z=0 \quad \text{at} \quad z=0, \tag{1.2}$$

where $K=\sigma^2/g$ and g is the gravitational acceleration, and

$$\left.\begin{array}{ll} & \partial\phi/\partial z=0 \quad \text{at} \quad z=H \quad (\text{finite depth } H)\\ \text{or} & \phi\to 0 \quad \text{as} \quad z\to\infty \quad (\text{infinite depth}).\end{array}\right\} \tag{1.3}$$

Conditions on ϕ on the body and as $R\to\infty$ are necessary to determine the motion and will involve either the scattering of a plane wave or the radiation of waves by the body.

Recently, Newman (1975) has shown how in the corresponding two-dimensional problem there is a relation between the reflexion and transmission coefficients R and T and the amplitudes of the waves generated in either direction by forced oscillations of the body. It is obtained by strategic use of Green's theorem and when the cylinder is symmetric about a vertical plane, reduces to simple expressions for R and T in terms of the phases of waves generated by symmetric and antisymmetric oscillations of the body. Then follows the deduction that the latter are independent of the type of oscillation.

Since the solution for high frequency plane surface waves incident on a sphere submerged in infinitely deep water (Davis 1974*a*) showed features similar to those of the corresponding solution for a circular cylinder (Davis 1974*b*), it seems reasonable to apply Newman's ideas to the three-dimensional case. A plane wave travelling from the $\theta = \pi$ direction has potential of the form, with $K = k \tanh kH$,

$$\phi = \frac{\cosh k\,(H-z)}{\cosh kH} e^{ikR\cos\theta} = \frac{\cosh k\,(H-z)}{\cosh kH} \sum_{m=0}^{\infty} \epsilon_m\, i^m J_m\,(kR) \cos m\,\theta$$

(Abramovitz & Stegun 1964, § 9.1.21), where ϵ_m is Neumann's symbol ($\epsilon_0 = 1$, $\epsilon_m = 2$ when $m \geqslant 1$). When $H \to \infty$, the ratio outside the summation reduces to e^{-Kz}. Each Bessel function can be expressed as an average of Hankel functions and then it is seen that the plane wave is a superposition of incoming and outgoing cylindrical waves of equal amplitude. The scattering effect of the body on a plane wave and the radiative effect of the body when oscillating both produce outgoing cylindrical waves at infinity. It is of considerable advantage in this three-dimensional situation that the plane wave can be allowed to approach the body from any direction. In the axisymmetric case, results of the same type as Newman's are obtained and illustrated by the example of a sphere. In the general case, corresponding results are not obtained because there is interchange of energy between Fourier modes.

2. The axisymmetric body

Here the z axis is an axis of symmetry in the immersed body. When the plane wave is incident from the direction $\theta = \psi + \pi$, the total scattered-wave potential $\phi_s(R, \theta, z; \psi)$ is such that

$$\partial\phi_s/\partial n = 0 \quad \text{on the body,} \tag{2.1}$$

$$\phi_s(\psi) \sim \frac{\cosh k(H-z)}{2\cosh kH} \sum_{m=0}^{\infty} \epsilon_m\, i^m\, [(1+a_m) H_m^{(1)}(kR) + H_m^{(2)}(kR)] \cos m(\theta-\psi) \quad \text{as} \quad R\to\infty. \tag{2.2}$$

The symmetry demands that, like the plane wave, only even functions of $\theta - \psi$ enter and with coefficients independent of ψ.

The radiation potential $\phi_r(R, \theta, z)$ satisfies

$$\partial\phi_r/\partial n = f \quad \text{on the body,} \tag{2.3}$$

where f is a prescribed real-valued function of position on the body, and is such that

$$\phi_r \sim \frac{\cosh k(H-z)}{2\cosh kH} \sum_{m=0}^{\infty} \epsilon_m (A_m \cos m\theta + B_m \sin m\theta)\, H_m^{(1)}(kR) \quad \text{as} \quad R\to\infty. \tag{2.4}$$

When Green's theorem is applied to $\phi_s(\psi_1)$ and $\overline{\phi_s(\psi_2)}$ throughout the fluid region, (1.1)–(1.3) and (2.1) ensure that the only non-zero contribution arises from infinity and (2.2) then yields

$$\sum_{m=0}^{\infty} \epsilon_m (|1+a_m|^2 - 1) \cos m\,(\psi_1 - \psi_2) = 0.$$

Since this is valid for all ψ_1 and ψ_2, it follows that

$$|1+a_m| = 1 \quad (m \geqslant 0). \tag{2.5}$$

When Green's theorem is applied to $\phi_r - \overline{\phi_r}$ and $\phi_s(\psi)$ throughout the fluid region, then, again, since $\phi_r - \overline{\phi_r}$ satisfies (2.1) because f in (2.3) is real, the only non-zero contribution arises from infinity and (2.2) and (2.4) imply that

$$\sum_{m=0}^{\infty} \epsilon_m i^m \left[A_m \cos m\psi + B_m \sin m\psi + (1+a_m)\left(\overline{A_m} \cos m\psi + \overline{B_m} \sin m\psi\right)\right] = 0.$$

Since this is true for all ψ, it follows that

$$\left.\begin{aligned} A_m + (1+a_m)\overline{A_m} &= 0 \quad (m \geqslant 0), \\ B_m + (1+a_m)\overline{B_m} &= 0 \quad (m \geqslant 1). \end{aligned}\right\} \tag{2.6}$$

Instead of receiving the plane wave from all possible directions, (2.6) could have been obtained by observing that, owing to the symmetry, each A_m and B_m is determined only by the corresponding Fourier component in f, each of which is a function of position on the cross-section of the boundary of the body. Also, (2.5) can be deduced from (2.6) by writing

$$A_m = |A_m| \exp(i\delta_m), \quad B_m = |B_m| \exp(i\nu_m). \tag{2.7}$$

Of interest now is that (2.6) implies that

$$1 + a_m = -\exp(2i\delta_m), \quad \delta_m = \pm\tfrac{1}{2}\pi + \tfrac{1}{2}\arg(1+a_m), \tag{2.8}$$

and similarly for ν_m. Alternatively

$$a_m = -2\exp(i\delta_m)\cos\delta_m = -2\exp(i\nu_m)\cos\nu_m,$$

i.e.

$$|\cos\delta_m| = |\cos\nu_m| = \tfrac{1}{2}|a_m|, \tag{2.9}$$

$$\delta_m, \nu_m = \arg a_m \quad \text{or} \quad \arg a_m + \pi.$$

Thus, subject to a possible shift of 180°, the cylindrical reflected wave in each mode arising from the incidence of the plane wave is, by comparison of (2.2) and (2.4), either in phase (with m even) or a quarter of a period behind (m odd) the radiated wave of the same mode. The latter therefore have phases independent of f in (2.3) and are determined solely by the shape and position of the body.

In the particular case discussed by Davis (1974*a*), the body is a sphere of radius a and mean depth h ($> a$) and Ka is large. Then, asymptotically it is found that

$$a_m \sim 2i\, e^{-2K(h-a)}\left[(\pi Ka)^{\frac{1}{2}} + O(1)\right].$$

Hence it can now be deduced from (2.8) that

$$\delta_m, \nu_m \sim \pm\tfrac{1}{2}\pi + e^{-2K(h-a)}\left[(\pi Ka)^{\frac{1}{2}} + O(1)\right]$$

for any given real-valued normal derivative on the sphere.

3. The arbitrary body

When the plane wave again travels towards the direction $\theta = \psi$, the total scattered-wave potential $\phi_s(R, \theta, z, \psi)$ satisfies (2.1) and is such that

$$\phi_s(\psi) \sim \frac{\cosh k(H-z)}{2\cosh kH} \sum_{m=0}^{\infty} \epsilon_m i^m \{[H_m^{(1)}(kR) + H_m^{(2)}(kR)] \cos m(\theta - \psi)$$
$$+ H_m^{(1)}(kR)[a_m(\psi) \cos m\theta + b_m(\psi) \sin m\theta]\} \quad \text{as} \quad R \to \infty. \quad (3.1)$$

Each coefficient $a_m(\psi)$ or $b_m(\psi)$ is periodic in ψ with period 2π and has a Fourier series which in general involves both sines and cosines. There is now, unlike the axisymmetric case, interchange of energy between modes characterized by m in the above summation. Instead of (2.6) an infinite set of homogeneous simultaneous equations is obtained with coefficients depending on $\{a_m(\psi), b_m(\psi)\}$, i.e. on the shape and position of the body.

When the body is a weak scatterer, the functions $a_m(\psi)$ and $b_m(\psi)$ are small and it then follows that A_m and B_m are almost pure imaginary. An obvious example of this situation, based on the submerged-sphere case mentioned previously, is when the wavelength is small compared with the depth of the highest point of a submerged body.

Equivalent results, in terms of Kochin functions, have been derived by Newman (1976).

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